

SEMIDEFINITELY REPRESENTABLE CONVEX SETS

Preliminary version

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ABSTRACT. We show that there are many (compact) convex semi-algebraic sets in euclidean space that do not have a semidefinite representation. This gives a negative answer to a question by Nemirovski, resp. it shows that the Helton-Nie conjecture is false.

INTRODUCTION

Semidefinite programming is a far-reaching generalization of linear programming. While a linear program optimizes a linear function over a polyhedron, a semidefinite program optimizes it over a convex region described by symmetric matrix inequalities. Using interior path methods, semidefinite programs can be solved efficiently and in polynomial time, up to any prescribed accuracy. They have numerous applications in applied mathematics, engineering, control theory and so forth.

The feasible regions of semidefinite programs are called semidefinitely representable sets, or briefly sdp sets. These are the sets $K \subseteq \mathbb{R}^n$ that can be written

$$K = \left\{ \xi \in \mathbb{R}^n : \exists \eta \in \mathbb{R}^m \ A + \sum_{i=1}^n \xi_i B_i + \sum_{j=1}^m \eta_j C_j \succeq 0 \right\}$$

where $m \geq 0$, A, B_i, C_j are real symmetric matrices of the same size and $M \succeq 0$ means that M is positive semidefinite. There has been considerable interest in characterizing sdp sets by their geometric properties. Nemirovski [16] in his 2006 plenary address at ICM Madrid remarked: “A seemingly interesting question is to characterize SDP-representable sets. Clearly, such a set is convex and semi-algebraic. Is the inverse also true? (...) This question seems to be completely open.” Helton and Nie ([10] p. 790) conjectured that the answer is in fact yes, i.e., that every convex semi-algebraic set in \mathbb{R}^n is semidefinitely representable.

Although the general question has so far been elusive, many results in support of the Helton-Nie conjecture have been obtained. The class of sdp sets is known to be closed under the operations of taking linear images or preimages, finite intersections, convex hulls of finite unions ([10], [20]), topological closures, and under convex duality respectively polarity. Helton and Nie ([10], [11]) gave a series of sufficient conditions for semidefinite representability of a convex semi-algebraic set K , in terms of curvature conditions for the boundary. Roughly, their results are saying that when K is compact and its boundary is sufficiently nonsingular and has strictly positive curvature, then K has a semidefinite representation. Netzer [18] proved that the interior of an sdp set is again an sdp set, and more generally, that removing suitably parametrized families of faces from an sdp set results again in an sdp set. Netzer and Sanyal [19] showed that smooth hyperbolicity cones are semidefinitely

representable, thereby proving a weak version of the generalized Lax conjecture. Scheiderer [28] showed that convex hulls of one-dimensional semi-algebraic sets are always semidefinitely representable. In particular, the Helton-Nie conjecture is true for subsets of the plane.

In addition there are plenty of further results on semidefinite representations for particular kinds of sets. See, for example, [5], [6], [8], [13], [21], [22], [27], and see [17], [2], [16], [3] (ch. 6) or [1] (ch. 2, 4 and 5) for surveys.

An important general technique for constructing semidefinite representations was introduced by Lasserre [14], and independently by Parrilo. It is based on a dual relaxation principle and is generally known as the moment relaxation technique. Starting with a (basic closed) semi-algebraic set S in \mathbb{R}^n , it produces outer approximations of the convex hull of S that are semidefinitely representable. Under favorable conditions, moment relaxation is known to become exact, meaning that a suitable such approximation coincides with the convex hull of S , up to taking closures.

In this paper we exhibit, for the first time, non-trivial conditions that are necessary for semidefinite representability. They are based on semidefinite duality and may be vaguely expressed by saying that there are no more (closed) sdp sets than those obtained from exact moment relaxation in a generalized sense. We then use arguments from algebraic geometry, in particular properties of smooth morphisms of varieties, to show that these conditions are indeed non-trivial, and to produce lots of concrete examples of convex sets without a semidefinite representation. Among them are natural prominent sets like the cone of psd forms of fixed degree in $\mathbb{R}[x_1, \dots, x_n]$, in every case where this cone is different from the sums of squares cone (Corollary 4.25). In fact, we prove for every semi-algebraic set $S \subseteq \mathbb{R}^n$ of dimension at least two that there exist polynomial maps $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^m$ for which the closed convex hull of $\varphi(S)$ in \mathbb{R}^m has no semidefinite representation. This is in marked contrast to the case where S has dimension one, when it is known that the convex hull of S is always semidefinitely representable, by the results of [28].

The paper is organized as follows. In Section 2 we recall and generalize the moment relaxation construction, arriving at general sufficient conditions for semidefinite representability. In Section 3 we show that the conditions obtained in Section 2 are also necessary. The main result is Theorem 3.14. In Section 4 we present concrete constructions of closed convex sets that violate the necessary conditions from Section 3, and we give a number of explicit examples.

There are many obvious questions that remain open at this point, and that we plan to study more systematically in the future. For example, what is the minimal dimension of a convex semi-algebraic set without semidefinite representation? The smallest dimension that we realize in this paper is 16, but we expect that the true answer should be much smaller. Is it three? Our necessary and sufficient condition for sdp representability is often hard to decide in concrete cases. What are alternative characterizations that are easier to work with?

1. PRELIMINARIES AND NOTATION

1.1. A symmetric matrix $A \in \text{Sym}_d(\mathbb{R})$ is said to be positive semidefinite, denoted $A \succeq 0$, if all its eigenvalues are nonnegative. If in addition all eigenvalues are nonzero then A is positive definite, written $A \succ 0$. The same terminology applies when the field \mathbb{R} of real numbers is replaced by a real closed field R .

1.2. A set $K \subseteq \mathbb{R}^n$ is called a *spectrahedron* if there exist $d \geq 1$ and $M_0, \dots, M_n \in \text{Sym}_d(\mathbb{R})$ such that $K = \{\xi \in \mathbb{R}^n : M_0 + \sum_{i=1}^n \xi_i M_i \succeq 0\}$. The set K is said to be *semidefinitely representable* (or an *sdp set* for short), if there exists a spectrahedron $S \subseteq \mathbb{R}^m$ for some m and a linear map $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that $K = f(S)$.

Spectrahedra have also been called LMI-sets or LMI-representable sets. In the plane, spectrahedra are characterized by the former Lax conjecture, which has been proved by Helton-Vinnikov [12] in 2007. In higher dimension there exist only conjectural characterizations of spectrahedra (so-called generalized Lax conjecture).

Semidefinitely representable sets have occurred under various different names as well, such as SDP-representable set, projected spectrahedra, lifted-LMI representable sets, or spectrahedral shadows. These sets are convex and semi-algebraic, but so far no other restrictions were known.

1.3. For V a vector space over a field k we denote the dual space of V by $V^\vee = \text{Hom}_k(V, k)$. Let V be a finite-dimensional \mathbb{R} -vector space. By a (convex) cone C in V we mean a non-empty set $C \subseteq V$ with $C + C \subseteq C$ and $aC \subseteq C$ for all real numbers $a \geq 0$. Given any set $M \subseteq V$ let $\text{conv}(M)$ denote the convex hull of M , and let $\text{cone}(M)$ be the convex cone generated by M (consisting of all finite linear combinations of elements of M with non-negative coefficients). Moreover, $M^* \subseteq V^\vee = \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$ denotes the (closed convex) cone dual to M , i.e. $M^* = \{\lambda \in V^\vee : \forall x \in M \lambda(x) \geq 0\}$. When M is a semi-algebraic set then so are $\text{conv}(M)$ and $\text{cone}(M)$, by Carathéodory's lemma, and also M^* .

1.4. Given a field k , a k -algebra is a (commutative) ring containing k . The ideal in a ring A generated by elements $a_i \in A$ ($i \in I$) is denoted $\langle a_i : i \in I \rangle$. We use the language of schemes to speak about algebraic varieties. Algebraic k -varieties are assumed to be reduced, but not necessarily irreducible. Therefore if V is an affine algebraic k -variety, then V is the spectrum of a reduced finitely generated k -algebra. The affine coordinate ring of V is written $k[V] = \Gamma(V, \mathcal{O}_V)$, as usual. The set of k -rational points of V is $V(k)$. For $\xi \in V(k)$, the local ring of V at ξ is denoted $\mathcal{O}_{V, \xi}$. By $\mathfrak{m}_{V, \xi}$ we invariantly denote both the maximal ideal of $\mathcal{O}_{V, \xi}$ and its preimage in $k[V]$. Given a morphism $\phi : X \rightarrow Y$ of affine k -varieties, the associated homomorphism $k[Y] \rightarrow k[X]$ of k -algebras is denoted ϕ^* . Given a field extension K/k we write $X_K := X \times_k \text{Spec}(K)$ for the base field extension of X , and similarly $\phi_K : X_K \rightarrow Y_K$ for the base field extension of ϕ . Moreover we write $K[X] = k[X] \otimes_k K$, and $\phi_K^* = \phi^* \otimes 1$ denotes the induced map $K[Y] \rightarrow K[X]$.

1.5. Let V be an affine k -variety, and let $L \subseteq k[V]$ be a k -linear subspace of finite dimension. Let $\mathbf{S}^\bullet L = \bigoplus_{d \geq 0} \mathbf{S}^d L$ be the symmetric algebra over L , and let $\mathbf{S}^\bullet L \rightarrow k[V]$ be the natural k -homomorphism induced by the inclusion $L \subseteq k[V]$. The associated morphism of affine k -varieties will be denoted $\varphi_L : V \rightarrow \mathbb{A}_L$, where $\mathbb{A}_L := \text{Spec}(\mathbf{S}^\bullet L)$ is the affine space with coordinate ring $\mathbf{S}^\bullet L$. In plainer terms, upon fixing a linear basis g_1, \dots, g_m of L , we may identify φ_L with the map $V \rightarrow \mathbb{A}^m$ given by $\xi \mapsto (g_1(\xi), \dots, g_m(\xi))$. Note that $\mathbb{A}_L(k) = L^\vee = \text{Hom}_k(L, k)$, the linear space dual to L .

2. SUFFICIENT CONDITIONS FOR SEMIDEFINITE REPRESENTABILITY

Let V be an affine \mathbb{R} -variety over \mathbb{R} , let $S \subseteq V(\mathbb{R})$ be a semi-algebraic set. By $\mathcal{P}(S) \subseteq \mathbb{R}[V]$ we denote the saturated preordering associated to S , i.e. $\mathcal{P}(S) = \{p \in \mathbb{R}[V] : f \geq 0 \text{ on } S\}$.

2.1. We start by informally recalling the moment relaxation construction, due to Lasserre [14] and independently Parrilo. Let $L \subseteq \mathbb{R}[V]$ be a linear subspace with $\dim(L) = m < \infty$, and let $\varphi_L: V \rightarrow \mathbb{A}_L \cong \mathbb{A}^m$ be the associated morphism. Assume that $S \subseteq V(\mathbb{R})$ is a basic closed semi-algebraic set, say $S = \{\xi \in V(\mathbb{R}) : h_i(\xi) \geq 0 \text{ (} i = 1, \dots, r \text{)}\}$ where $h_1, \dots, h_r \in \mathbb{R}[V]$. We are trying to find an sdp representation of the convex hull K of $\varphi_L(S)$ in $\mathbb{A}_L(\mathbb{R}) = L^\vee \cong \mathbb{R}^m$, or at least an approximate such representation.

For this fix a sequence W_0, \dots, W_r of finite-dimensional linear subspaces of $\mathbb{R}[V]$. Any $f \in L_1 := \mathbb{R}1 + L \subseteq \mathbb{R}[V]$ that has a representation $f = s_0 + \sum_{i=1}^r s_i h_i$ with s_i a sum of squares of elements of W_i ($i = 0, \dots, r$) is obviously nonnegative on S . So the set of all such f is a convex cone C , contained in $L_1 \cap \mathcal{P}(S)$. By construction, the dual cone $C^* \subseteq L_1^\vee$ is an sdp cone in an explicit way. Via the obvious embedding $\mathbb{A}_L(\mathbb{R}) = L^\vee \subseteq L_1^\vee$ it gives an sdp set K' in $\mathbb{A}_L(\mathbb{R})$ that contains K . Enlarging the spaces W_i , or adding more inequalities h_i to the description of S , results in K' getting smaller, and therefore becoming a closer approximation to K . Of particular interest is the case where $C = L_1 \cap \mathcal{P}(S)$. This condition is usually rephrased by saying that the linear polynomials nonnegative on $\varphi_L(S)$ (i.e. the elements of $L_1 \cap \mathcal{P}(S)$) have weighted sum of squares representations of uniformly bounded degrees (the “degree bounds” being given by the subspaces W_i). The moment relaxation is *exact* in this case, in the sense that K' is sandwiched between K and \overline{K} . Therefore, under this assumption and up to taking closures, K coincides with the sdp set K' arising from the construction. It is well known that this implies that the closure \overline{K} itself is an sdp set.

We now generalize this procedure, to arrive at a general sufficient condition for semidefinite representability. First two auxiliary lemmas.

Lemma 2.2. *Let V be an affine \mathbb{R} -variety, and let $S \subseteq V(\mathbb{R})$ be a semi-algebraic set. Let $L \subseteq \mathbb{R}[V]$ be a finite-dimensional linear subspace, and write $L_1 := \mathbb{R}1 + L$.*

- (a) *$\varphi_L(S)$ is a semi-algebraic subset of L^\vee .*
- (b) *The closed convex hull $\overline{\text{conv}(\varphi_L(S))}$ of $\varphi_L(S)$ in L^\vee consists of all $\lambda \in L^\vee$ that satisfy $\lambda(g) \geq 0$ for every $g \in L_1 \cap \mathcal{P}(S)$.*

Proof. (a) follows from the Tarski-Seidenberg theorem, and (b) is a consequence of convex duality. \square

Lemma 2.3. *Let A be an \mathbb{R} -algebra, let $U \subseteq A$ be a linear subspace with $\dim(U) < \infty$, and let UU be the linear subspace of A spanned by all products uu' ($u, u' \in U$). Then ΣU^2 , the set of all sums of squares of elements of U , is an sdp cone in UU .*

Proof. Choose a linear basis u_1, \dots, u_n of U . The linear map $f: \text{Sym}_n(\mathbb{R}) \rightarrow UU$, $(a_{ij}) \mapsto \sum_{i,j} a_{ij} u_i u_j$ satisfies $\Sigma U^2 = f(\text{Sym}_n^+(\mathbb{R}))$, which shows the claim. \square

We keep fixing an affine \mathbb{R} -variety V , a semi-algebraic set $S \subseteq V(\mathbb{R})$ and a finite-dimensional linear subspace $L \subseteq \mathbb{R}[V]$. As before write $L_1 = L + \mathbb{R}1$.

Proposition 2.4. *Let $\phi_i: X_i \rightarrow V$ ($i = 1, \dots, m$) be finitely many morphisms of affine \mathbb{R} -varieties. For every $i = 1, \dots, m$ let $U_i \subseteq \mathbb{R}[X_i]$ be a finite-dimensional linear subspace, and assume that the following two properties hold:*

- (1) *$S \subseteq \phi_i(X_i(\mathbb{R}))$ for $i = 1, \dots, m$;*
- (2) *for every $f \in L_1 \cap \mathcal{P}(S)$ there exists $i \in \{1, \dots, m\}$ such that $\phi_i^*(f) \in \mathbb{R}[X_i]$ is a sum of squares of elements of U_i (in $\mathbb{R}[X_i]$).*

Then $\overline{\text{conv}(\varphi_L(S))}$, the closed convex hull of $\varphi_L(S)$ in L^\vee , has a semidefinite representation.

Proof. Write $C := L_1 \cap \mathcal{P}(S)$, which is a closed convex cone in L_1 . For a given index $i \in \{1, \dots, m\}$ let $C_i \subseteq L_1$ be the cone of all $f \in L_1$ for which $\phi_i^*(f)$ is a sum of squares of elements of U_i in $\mathbb{R}[X_i]$. By Lemma 2.3, and since linear preimages of sdp sets are again sdp sets, C_i is an sdp cone in L_1 . By condition (1), elements of C_i are non-negative on S , which means $C_i \subseteq C$. Therefore $C = \bigcup_{i=1}^m C_i$ by (2), and hence we have $C^* = \bigcap_{i=1}^m C_i^*$ for the dual cones. For every index i the cone C_i^* , being the dual cone to an sdp cone, is itself an sdp cone. So it follows that C^* is an sdp cone in L_1^\vee .

For the convex hull $K := \text{conv}(\varphi_L(S)) \subseteq L^\vee$ we have $\overline{K} = \{\lambda \in L^\vee : \forall g \in C \lambda(g) \geq 0\}$ (Lemma 2.2). This says that, under the restriction map $\rho: L_1^\vee \rightarrow L^\vee$ dual to the inclusion $L \subseteq L_1$, we have $\overline{K} = \rho(C^*)$. So \overline{K} is an sdp set, as asserted. \square

Remarks 2.5.

1. Proposition 2.4 can be seen as a generalization of the moment relaxation construction. To explain this, assume we are in the situation of 2.1. Let X be the affine \mathbb{R} -variety obtained by formally adjoining square roots of h_1, \dots, h_r to $\mathbb{R}[V]$, i.e. $\mathbb{R}[X] = \mathbb{R}[V][t_1, \dots, t_r] / (t_i^2 - h_i, i = 1, \dots, r)$, and let $\phi: X \rightarrow V$ be the natural map. Then clearly $\phi(X(\mathbb{R})) = S$. If subspaces $W_i \subseteq \mathbb{R}[V]$ as in 2.1 have been found such that the sufficient exactness condition from 2.1 is satisfied, i.e. if $L_1 \cap \mathcal{P}(S) = C$ (in the notation of 2.1), this implies that for every $f \in L_1 \cap \mathcal{P}(S)$ the pull-back $\phi^*(f) \in \mathbb{R}[X]$ is a sum of squares of elements from the subspace $U := \phi^*(W_0) + \sum_{i=1}^r \phi^*(W_i)\sqrt{h_i}$ in $\mathbb{R}[X]$. So under this assumption, the conditions of Proposition 2.4 are satisfied with $m = 1$ and these particular choices of ϕ and U .

2. Conversely, the construction in Proposition 2.4 of an sdp representation under the assumptions made there is done essentially by reduction to a construction of moment relaxation type. We leave it to the reader to make this statement precise.

3. The proof of Proposition 2.4 is constructive in the following sense. If the morphisms $\phi_i: X_i \rightarrow V$ as well as the linear subspaces $U_i \subseteq \mathbb{R}[X_i]$ are given explicitly, we can deduce an explicit semidefinite representation of $\overline{\text{conv}(\varphi_L(S))}$ from this data.

Example 2.6. Let C be a nonsingular affine curve over \mathbb{R} for which $C(\mathbb{R})$ is compact. Let $L \subseteq \mathbb{R}[C]$ be a finite-dimensional linear subspace, and consider the associated map $\varphi_L: C(\mathbb{R}) \rightarrow \mathbb{A}_L(\mathbb{R}) = L^\vee$. By the main result of [28] (see e.g. Corollary 4.4), there exists a finite-dimensional linear subspace $U \subseteq \mathbb{R}[C]$ such that every $f \in L + \mathbb{R}1$ that is nonnegative on $C(\mathbb{R})$ is a sum of squares of elements from U . Using this fact, Proposition 2.4 applies with $m = 1$ and $\phi_1: X_1 \rightarrow C$ the identity map of C , showing that the convex hull of $\varphi_L(C(\mathbb{R}))$ in L^\vee is semidefinitely representable. (This consequence was already drawn in [28].)

Remark 2.7. Later (Remark 3.17 below) we'll see that it is not enough in Proposition 2.4 to replace condition (2) by the weaker condition that every $f \in L_1 \cap \mathcal{P}(S)$ becomes a sum of squares in one of the $\mathbb{R}[X_i]$. Rather, it is essential that such sos representations exist in a uniform way.

3. NECESSARY CONDITIONS FOR SEMIDEFINITE REPRESENTABILITY

In the previous section we stated sufficient conditions for semidefinite representability. We now show that these conditions are also necessary.

3.1. We'll have to work not only over the field \mathbb{R} of real numbers, but also over real closed extension fields R, R' of \mathbb{R} . Given a semi-algebraic set $M \subseteq \mathbb{R}^n$, the base field extension of M to R is denoted M_R (see [4] 5.1). Also recall that we write

$$M^* = \left\{ u \in \mathbb{R}^n : \forall \xi \in M \sum_{i=1}^n u_i \xi_i \geq 0 \right\}.$$

This is a closed convex semi-algebraic cone in \mathbb{R}^n . Extending M^* to R gives $(M^*)_R = \{ u \in R^n : \forall \xi \in M_R \sum_{i=1}^n u_i \xi_i \geq 0 \}$. We simply write M_R^* for this set.

We keep writing $x = (x_1, \dots, x_n)$, and start by recalling one form of duality in semidefinite programming (see [23]):

Proposition 3.2. *Let $M_1, \dots, M_n \in \text{Sym}_d(\mathbb{R})$, write $M(\xi) = \sum_{i=1}^n \xi_i M_i$ for $\xi \in \mathbb{R}^n$, and let $C = \{ \xi \in \mathbb{R}^n : M(\xi) \succeq 0 \}$ be the associated spectrahedral cone. Assume that $M(\xi^0) \succ 0$ for some $\xi^0 \in S$. Then the dual cone of C has the following semidefinite representation:*

$$C^* = \left\{ \left(\langle B, M_1 \rangle, \dots, \langle B, M_n \rangle \right) : B \in \text{Sym}_d(\mathbb{R}), B \succeq 0 \right\} \subseteq \mathbb{R}^n. \quad \square$$

Remark 3.3. By the transfer principle of Tarski-Seidenberg, the analogue of 3.2 holds over any real closed field R . In particular, if $M_1, \dots, M_n \in \text{Sym}_d(R)$ and $b \in R^n$ are such that $\sum_{i=1}^n b_i \xi_i \geq 0$ for every $\xi \in R^n$ with $M(\xi) := \sum_{i=1}^n \xi_i M_i \succeq 0$, and if $M(\xi^0) \succ 0$ for some $\xi^0 \in R^n$, there exists $B \in \text{Sym}_d(R)$ with $B \succeq 0$ and $b_i = \langle B, M_i \rangle$ for $i = 1, \dots, n$.

3.4. Given $A, B \in \text{Sym}_d(\mathbb{R})$ let $\langle A, B \rangle = \text{tr}(AB)$. More generally, let R, R' be real closed extension fields of \mathbb{R} . Given $A = (a_{ij}) \in \text{Sym}_d(R')$ and $B = (b_{ij}) \in \text{Sym}_d(R)$, we put

$$\langle A, B \rangle^\otimes := \sum_{i,j=1}^d a_{ij} \otimes b_{ij},$$

considered as an element of $R' \otimes R$ (tensor product over \mathbb{R}). Similarly, for $u \in R'^n$ and $v \in R^n$ let

$$\langle u, v \rangle^\otimes := \sum_{i=1}^n u_i \otimes v_i \in R' \otimes R.$$

Proposition 3.5. *Let $C \subseteq \mathbb{R}^n$ be an sdp cone, let $a \in C_{R'} \subseteq R'^n$ and $b \in C_R^* \subseteq R^n$ (see 3.1). Then $\langle a, b \rangle^\otimes$ is a sum of squares in $R' \otimes R$.*

Proof. There are $d \geq 1$, $m \geq 0$ and linear matrix pencils $M(x) = \sum_{i=1}^n x_i M_i$, $N(y) = \sum_{j=1}^m y_j N_j$ in $\text{Sym}_d(\mathbb{R})$ (with $y = (y_1, \dots, y_m)$) such that, writing

$$S = \{ (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m : M(\xi) + N(\eta) \succeq 0 \},$$

C is the projection of S to the first n coordinates. Hence there is $a' \in R'^m$ such that $(a, a') \in S_{R'}$, and moreover $(b, 0) \in S_R^* \subseteq R^n \times R^m$. Since $\langle a, b \rangle^\otimes = \langle (a, a'), (b, 0) \rangle^\otimes$, we see that it suffices to prove the assertion for S instead of C .

So we may assume that C is a spectrahedral cone. One easily reduces to the case where C has non-empty interior in \mathbb{R}^n . Then C can be represented by a

homogeneous linear matrix inequality that is strictly feasible. This means, there are $d \geq 1$ and a linear system $M(x) = \sum_{i=1}^n x_i M_i$ in $\text{Sym}_d(\mathbb{R})$ such that $C = \{\xi \in \mathbb{R}^n : M(\xi) \succeq 0\}$, and such that $M(\xi^0) \succ 0$ for some $\xi^0 \in \mathbb{R}^n$. By Remark 3.3 there exists $B \in \text{Sym}_d(R)$ such that $B \succeq 0$ and $b_i = \langle M_i, B \rangle$ for $i = 1, \dots, n$. Therefore

$$\langle a, b \rangle^\otimes = \sum_{i=1}^n a_i \otimes \langle M_i, B \rangle = \left\langle \sum_{i=1}^n a_i M_i, B \right\rangle^\otimes = \langle M(a), B \rangle^\otimes.$$

(The middle equality is correct since the M_i have coefficients in \mathbb{R} .) We have $M(a) \succeq 0$ since $a \in C_{R'}$. Decomposing both psd matrices $M(a)$ and B as sums of psd symmetric rank one matrices (over R' , resp. over R) we get the claim, using that

$$\langle uu^t, vv^t \rangle^\otimes = \sum_{i,j=1}^d (u_i u_j) \otimes (v_i v_j) = \left(\sum_{i=1}^d u_i \otimes v_i \right)^2$$

is a single square in $R' \otimes R$, for $u \in R'^d$ and $v \in R^d$. \square

Inspecting the last proof we can deduce the following more precise information from it:

Corollary 3.6. *In the situation of Proposition 3.5, if C is presented by a projected linear matrix inequality of symmetric $d \times d$ matrices, then $\langle a, b \rangle^\otimes$ is a sum of d^2 squares in $R' \otimes R$.* \square

In the sequel we will consider the following

3.7. Assumption: $S \subseteq \mathbb{R}^n$ is a semi-algebraic set for which the closed convex cone $\text{cone}(S)$ generated by S in \mathbb{R}^n has a semidefinite representation.

From 3.5 we infer:

Corollary 3.8. *Let $S \subseteq \mathbb{R}^n$ satisfy Assumption 3.7, let R, R' be real closed field extensions of \mathbb{R} , and let $a \in S_{R'}$, $b \in S_R^*$. Then $\langle a, b \rangle^\otimes$ is a sum of squares in $R' \otimes R$.* \square

3.9. We keep the hypotheses of Corollary 3.8. There exists a finite-dimensional \mathbb{R} -linear subspace U of R' such that $\langle a, b \rangle^\otimes$ is a sum of squares of elements from $U \otimes R$ (inside $R' \otimes R$). Let A be the \mathbb{R} -subalgebra of R' generated by a_1, \dots, a_n and by U . Then $\langle a, b \rangle^\otimes$ is a sum of squares in the finitely generated R -algebra $A \otimes R$.

Lemma 3.10. *Let $S \subseteq \mathbb{R}^n$ satisfy Assumption 3.7, let R be a real closed field extension of \mathbb{R} , and let $b \in S_R^*$. Given any real closed field extension R' of \mathbb{R} and any $a \in S_{R'}$, there exists a morphism $\phi: X \rightarrow \mathbb{A}^n$ of affine \mathbb{R} -varieties satisfying $a \in \text{im}(\phi: X(R') \rightarrow R'^n)$ and such that $\sum_{i=1}^n \phi^*(x_i) \otimes b_i$ is a sum of squares in $\mathbb{R}[X] \otimes R = R[X]$.*

Proof. Let $A \subseteq R'$ be the finitely generated \mathbb{R} -subalgebra constructed in 3.9, and let $X = \text{Spec}(A)$ be the associated affine \mathbb{R} -variety. Write $x = (x_1, \dots, x_n)$, and let $\varphi: \mathbb{R}[x] \rightarrow A$ be the homomorphism of \mathbb{R} -algebras defined by $\varphi(x_i) = a_i$ ($i = 1, \dots, n$). Let $\phi: X \rightarrow \mathbb{A}^n$ be the associated morphism of varieties (satisfying $\phi^* = \varphi$). From the commutative triangle

$$\begin{array}{ccc} \mathbb{R}[x] & \xrightarrow{\varphi} & A \\ & \searrow a & \swarrow \text{incl} \\ & R' & \end{array}$$

we see $a \in \text{im}(\phi: X(R') \rightarrow R'^n)$. On the other hand,

$$\sum_{i=1}^n \phi^*(x_i) \otimes b_i = \sum_{i=1}^n a_i \otimes b_i = \langle a, b \rangle^{\otimes}$$

is a sum of squares in $A \otimes R = \mathbb{R}[X] \otimes R$, by the construction in 3.9. \square

Lemma 3.11. *Let $S \subseteq \mathbb{R}^n$ satisfy Assumption 3.7, let R be a real closed field extension of \mathbb{R} , and let $b \in S_R^*$. There exists a morphism $\phi: X \rightarrow \mathbb{A}^n$ of affine \mathbb{R} -varieties with $S \subseteq \text{im}(\phi: X(\mathbb{R}) \rightarrow \mathbb{R}^n)$, and such that $\sum_{i=1}^n \phi^*(x_i) \otimes b_i$ is a sum of squares in $\mathbb{R}[X] \otimes R = R[X]$.*

Proof. We use the real spectrum and the operation tilda, see e.g. [4] 7.2. Fix a point $\alpha \in \widetilde{S} \subseteq \text{Sper } \mathbb{R}[x]$, represent α by a homomorphism $\varphi: \mathbb{R}[x] \rightarrow R'$ into some real closed field R' , and put $a := (\varphi(x_1), \dots, \varphi(x_n)) \in R'^n$. Then $a \in S_{R'}$. So by Lemma 3.10 there exists a morphism $\phi: X \rightarrow \mathbb{A}^n$ of affine \mathbb{R} -varieties for which $a \in \phi(X(R'))$, and such that $\sum_{i=1}^n \phi^*(x_i) \otimes b_i$ is a sum of squares in $\mathbb{R}[X] \otimes R$. The first condition means $\alpha \in \phi(X(\mathbb{R}))$. By compactness of the “spectral” topology on the real spectrum ([4] 7.1.12), it follows that there exist finitely many morphisms $\phi_k: X_k \rightarrow \mathbb{A}^n$ ($k = 1, \dots, r$) of affine \mathbb{R} -varieties such that $\sum_{i=1}^n \phi_k^*(x_i) \otimes b_i$ is a sum of squares in $\mathbb{R}[X_k] \otimes R$ for every index k , and such that S is contained in $\bigcup_{k=1}^r \text{im}(\phi_k: X_k(\mathbb{R}) \rightarrow \mathbb{R}^n)$. Let X be the disjoint sum of the X_k (so $\mathbb{R}[X] = \mathbb{R}[X_1] \times \dots \times \mathbb{R}[X_r]$), and let $\phi: X \rightarrow \mathbb{A}^n$ be the morphism with $\phi|_{X_k} = \phi_k$ for $k = 1, \dots, r$. Then $\phi: X \rightarrow \mathbb{A}^n$ has the required properties. \square

Proposition 3.12. *Let $S \subseteq \mathbb{R}^n$ satisfy Assumption 3.7. There exists a morphism $\phi: X \rightarrow \mathbb{A}^n$ of affine \mathbb{R} -varieties, together with a finite-dimensional \mathbb{R} -linear subspace $U \subseteq \mathbb{R}[X]$, such that $S \subseteq \phi(X(\mathbb{R}))$, and such that the following holds: For every $v \in S^* \subseteq \mathbb{R}^n$, the element $\phi^*(\sum_{i=1}^n v_i x_i)$ in $\mathbb{R}[X]$ is a sum of squares of elements of U .*

Proof. We start with the following remark. Let $\psi: Y \rightarrow \mathbb{A}^n$ be any morphism of affine \mathbb{R} -varieties, and let $V \subseteq \mathbb{R}[Y]$ be any finite-dimensional \mathbb{R} -linear subspace. Consider the subset

$$M(\psi, V) := \left\{ v \in \mathbb{R}^n : \psi^* \left(\sum_{i=1}^n v_i x_i \right) \in \Sigma V^2 \text{ (in } \mathbb{R}[Y]) \right\}$$

of \mathbb{R}^n , where ΣV^2 denotes the subset of $\mathbb{R}[Y]$ consisting of the sums of squares of elements from V . Then $M(\psi, V)$ is a semi-algebraic set (and in fact an sdp cone, by Lemma 2.3). We now use a similar compactness argument as in the proof of 3.11, this time for S^* instead of S : Let $\beta \in \widetilde{S}^*$, and represent β by $b \in S_R^*$ where $R \supseteq \mathbb{R}$ is real closed. By Lemma 3.11 there are a morphism $\psi: Y \rightarrow \mathbb{A}^n$ of affine \mathbb{R} -varieties with $S \subseteq \psi(Y(\mathbb{R}))$, and a finite-dimensional \mathbb{R} -subspace $V \subseteq \mathbb{R}[Y]$, such that $\sum_i \psi^*(x_i) \otimes b_i$ is a sum of squares of elements from $V \otimes R$ (in $R[Y]$). Hence $b \in M(\psi, V)_R$, which means $\beta \in \widetilde{M(\psi, V)}$. By compactness, there exist finitely many morphisms $\phi_j: X_j \rightarrow \mathbb{A}^n$ of affine \mathbb{R} -morphisms, together with finite-dimensional \mathbb{R} -linear subspaces $U_j \subseteq \mathbb{R}[X_j]$ ($j = 1, \dots, s$), such that $S \subseteq \text{im}(\phi_j: X_j(\mathbb{R}) \rightarrow \mathbb{R}^n)$ for $j = 1, \dots, s$, and such that $S^* \subseteq \bigcup_{j=1}^s M(\phi_j, U_j)$.

Let X be the fibre product of X_1, \dots, X_s over \mathbb{A}^n , let $\phi: X \rightarrow \mathbb{A}^n$ be the obvious morphism, and let $\pi_j: X \rightarrow X_j$ be the j -th projection ($j = 1, \dots, s$).

Clearly we have $S \subseteq \phi(X(\mathbb{R}))$. The coordinate ring $\mathbb{R}[X]$ is the tensor product of $\mathbb{R}[X_1], \dots, \mathbb{R}[X_s]$ over $\mathbb{R}[x]$. Let $U = \sum_{j=1}^s \pi_j^*(U_j)$, an \mathbb{R} -linear subspace of $\mathbb{R}[X]$ with $\dim(U) < \infty$. Then it is clear that $S^* \subseteq M(\phi, U)$. \square

The following result is the combination of Propositions 2.4 and 3.12:

Theorem 3.13. *Let $S \subseteq \mathbb{R}^n$ be a semi-algebraic set, and let $C = \text{cone}(S)$ be the convex cone in \mathbb{R}^n generated by S . The closure \overline{C} has a semidefinite representation if and only if there exists a morphism $\phi: X \rightarrow \mathbb{A}^n$ of affine \mathbb{R} -varieties and an \mathbb{R} -linear subspace $U \subseteq \mathbb{R}[X]$ with $\dim(U) < \infty$ such that*

- (1) $S \subseteq \phi(X(\mathbb{R}))$,
- (2) for every $v \in S^* = C^*$, the element $\phi^*(\sum_{i=1}^n v_i x_i)$ of $\mathbb{R}[X]$ is a sum of squares of elements of U .

Proof. The second condition is necessary for \overline{C} to be an sdp cone by 3.12, and it is sufficient by 2.4. \square

Instead of working with convex cones we may also dehomogenize, and derive a non-homogeneous version of Theorem 3.13 from this theorem. Alternatively, we could as well have worked in an inhomogeneous setting from the beginning.

Theorem 3.14. *Let $S \subseteq \mathbb{R}^n$ be a semi-algebraic set, and let $K = \text{conv}(S)$ be the convex hull of S in \mathbb{R}^n . The closure \overline{K} has a semidefinite representation if and only if there exists a morphism $\phi: X \rightarrow \mathbb{A}^n$ of affine \mathbb{R} -varieties and an \mathbb{R} -linear subspace $U \subseteq \mathbb{R}[X]$ with $\dim(U) < \infty$ such that*

- (1) $S \subseteq \phi(X(\mathbb{R}))$,
- (2) for every linear polynomial $l \in \mathbb{R}[x]$ with $l \geq 0$ on S , the element $\phi^*(l)$ of $\mathbb{R}[X]$ is a sum of squares of elements of U .

Proof. If there exist ϕ and U satisfying (1) and (2), \overline{K} is an sdp set by Proposition 2.4. Conversely, assume that \overline{K} is an sdp set, and let $K_1 = \{1\} \times K \subseteq \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$. Since \overline{K}_1 is an sdp set, it is easy to see that $\text{cone}(\overline{K}_1)$ is an sdp cone in \mathbb{R}^{n+1} . Hence the closure of $\text{cone}(\overline{K}_1)$ is an sdp cone as well. Clearly, this last cone coincides with \overline{C}_1 , where C_1 is the convex cone in \mathbb{R}^{n+1} generated by $S_1 = \{1\} \times S$. Now we can apply the “only if” part of Theorem 3.13 to S_1 and C_1 and deduce the second condition in Theorem 3.14. \square

Remark 3.15. In 3.14 we may sharpen conditions (1) and (2) further. Assume we are given a morphism $\phi: X \rightarrow \mathbb{A}^n$ and a linear subspace $U \subseteq \mathbb{R}[X]$ as in 3.14. From (1) we deduce that there exists a semi-algebraic set $M \subseteq X(\mathbb{R})$ with $\phi(M) = S$ and with $\dim(M) = \dim(S)$. Let X' be the reduced Zariski closure of M in X , and let $U' \subseteq \mathbb{R}[X']$ be the image of U under $\mathbb{R}[X] \rightarrow \mathbb{R}[X']$. Then (1) and (2) hold as well for the restriction $\phi': X' \rightarrow \mathbb{A}^n$ of ϕ and for U' . Therefore, we can achieve in addition that $\dim(X) = \dim(S)$. On the other hand, condition (1) can be replaced by either $\overline{K} \subseteq \phi(X(\mathbb{R}))$ (to make it seemingly stronger), or by $K \subseteq \text{conv}(\phi(X(\mathbb{R})))$ (to make it seemingly weaker).

The following version is essentially identical with the “only if” part of Theorem 3.14. But depending on the context, it is sometimes more convenient to apply:

Let V be an affine \mathbb{R} -variety, and let $L \subseteq \mathbb{R}[V]$ be a finite-dimensional linear subspace. Let $\varphi_L: V \rightarrow \mathbb{A}^n \cong \mathbb{A}^{\dim(L)}$ ($n = \dim(L)$) be the associated morphism, see 1.5.

Corollary 3.16. *With V and L as above, let $S \subseteq V(\mathbb{R})$ be a semi-algebraic set. Assume that $\overline{\text{conv}(\varphi_L(S))}$, the closed convex hull in $\mathbb{A}_L(\mathbb{R}) = L^\vee$, is an sdp set. Then there exists a morphism $\phi: X \rightarrow V$ of affine \mathbb{R} -varieties, together with a finite-dimensional linear subspace $U \subseteq \mathbb{R}[X]$, such that $S \subseteq \phi(X(\mathbb{R}))$ and the following holds:*

For every real closed field $R \supseteq \mathbb{R}$ and every $f \in L_R + R1 \subseteq R[V]$ with $f \geq 0$ on S_R , the pull-back $\phi_R^(f)$ under $\phi_R: X_R \rightarrow V_R$ is a sum of squares of elements from $U \otimes R$ in $\mathbb{R}[X] \otimes R = R[X]$.*

(The converse is true as well, covered by Proposition 2.4.)

Proof. By 3.14 there exists a morphism $\psi: Y \rightarrow \mathbb{A}_L$ of affine \mathbb{R} -varieties, together with a finite-dimensional subspace $W \subseteq \mathbb{R}[Y]$, such that for every $f \in (\mathbb{R}1 + L) \cap \mathcal{P}(S)$ the pull-back $\psi^*(f) \in \mathbb{R}[Y]$ is a sum of squares of elements from W . Let X be the fibered product of V and Y over \mathbb{A}_L , let $\phi: X \rightarrow V$ be the canonical morphism. Then the condition in 3.16 is satisfied for $R = \mathbb{R}$. By Tarski-Seidenberg, it holds over an real closed extension R as well (see also the next remark). \square

Remark 3.17. Let $\phi: X \rightarrow V$ be a morphism of affine \mathbb{R} -varieties, let $L \subseteq \mathbb{R}[V]$ and $U \subseteq \mathbb{R}[X]$ be finite-dimensional linear subspaces, and let $S \subseteq V(\mathbb{R})$ be a semi-algebraic set. Assume that the following condition holds:

(*) For every $f \in L \cap \mathcal{P}_V(S)$, the pullback $\phi^*(f) \in \mathbb{R}[X]$ is a sum of squares of elements of U .

Then the extension of (*) to any real closed field extension R of \mathbb{R} holds as well. More precisely, any $f \in L_R = L \otimes R \subseteq R[V]$ with $f \geq 0$ on $S_R \subseteq V(R)$ becomes a sum of squares of elements of $U_R = U \otimes R$ in $R[X]$, by the Tarski principle.

In particular, any $f \in L \otimes R$ with $f \geq 0$ on S_R becomes a sum of squares in $R[X]$. We remark that this conclusion would fail in general if in (*) we had only required that $\phi^*(f)$ is a sum of squares in $\mathbb{R}[X]$. For instance, taking ϕ to be the identity of $X = V = \mathbb{A}^2$ and S the unit disk would give counter-examples, by the results of [25] and [26]. Rather, one needs that uniform sums of squares expressions exist as in (*), to guarantee that the condition is stable under real closed field extension.

Remark 3.18. It may not be obvious immediately, but the necessary condition for semidefinite representability found in Theorem 3.14 is quite restrictive. In particular, it allows to give plenty of examples of convex semi-algebraic sets that are not semidefinitely representable. In the next section we'll elaborate on this in more detail.

4. CONSTRUCTING EXAMPLES

We use properties of smooth morphisms of algebraic varieties, together with a weak version of generic smoothness, to construct examples of convex sets that have no semidefinite representation.

4.1. Let k be a field. Recall that a morphism $\phi: X \rightarrow Y$ of algebraic k -varieties is smooth at $x \in X$ if there exist affine open sets $U = \text{Spec}(A) \subseteq X$ and $V = \text{Spec}(B) \subseteq Y$ with $x \in U$ and $\phi(U) \subseteq V$ such that A is (via ϕ) B -isomorphic to $B[x_1, \dots, x_n]/(f_1, \dots, f_m)$, where $m \leq n$ and $\det(\partial f_i / \partial x_j)_{1 \leq i, j \leq m}$ is a unit in $\mathcal{O}_{X, x}$. It is equivalent that ϕ is flat at x and that the fibre $\phi^{-1}(\phi(x))$ is geometrically regular at x over the residue field of $\phi(x)$, see [7] 17.5.1. The smooth locus of ϕ , i.e. the set of points $x \in X$ at which ϕ is smooth, is Zariski open in X .

We will use the following weak version of generic smoothness (compare [9] Lemma III.10.5):

Proposition 4.2. *Let $\phi: X \rightarrow Y$ be a dominant morphism between irreducible k -varieties where $\text{char}(k) = 0$. Then there exists a non-empty Zariski open subset $U \subseteq X$ such that $\phi|_U: U \rightarrow Y$ is smooth.*

The following result is contained in [7] 17.5.3 as a particular case:

Proposition 4.3. *Let $\phi: X \rightarrow Y$ be a morphism of algebraic k -varieties. Let $\xi \in X(k)$, and write $A = \mathcal{O}_{X,\xi}$, $B = \mathcal{O}_{Y,\phi(\xi)}$. Then ϕ is smooth at ξ if and only if \hat{A} is isomorphic over \hat{B} to a power series algebra $\hat{B}[[t_1, \dots, t_m]]$.*

Of course, $\hat{}$ denotes completion of a local ring. From 4.3 we deduce the following observation:

Lemma 4.4. *Let $\phi: X \rightarrow Y$ be a morphism of algebraic k -varieties, and assume that ϕ is smooth at $\xi \in X(k)$. If $f \in \mathcal{O}_{Y,\phi(\xi)}$ is such that $\phi^*(f)$ is a sum of squares in $\hat{\mathcal{O}}_{X,\xi}$, then f is a sum of squares in $\hat{\mathcal{O}}_{Y,\phi(\xi)}$.*

Proof. Indeed, if an element of B becomes a sum of squares in $B[[t_1, \dots, t_m]]$, it is already a sum of squares in B . \square

4.5. We shall present two constructions. Each will give us concrete examples of non-sdp convex semi-algebraic sets. For both, the reasoning will be based on the following technical lemma. We will repeatedly assume that data is given as follows:

(*) V is an affine \mathbb{R} -variety, $L \subseteq \mathbb{R}[V]$ is a finite-dimensional linear subspace, $\varphi_L: V \rightarrow \mathbb{A}_L \cong \mathbb{A}^m$ ($m = \dim(L)$) is the associated morphism (see 1.5), and $S \subseteq V(\mathbb{R})$ is a semi-algebraic set. Moreover, V' is an irreducible component of V and $S' \subseteq S \cap V'(\mathbb{R})$ is a semi-algebraic set, Zariski-dense in V' .

Lemma 4.6. *Assume that (*) as in 4.5 is given. If $\overline{\text{conv}(\varphi_L(S))}$ is an sdp set in $\mathbb{A}_L(\mathbb{R})$, there exists a morphism $\psi: W \rightarrow V'$ of affine \mathbb{R} -varieties, together with $\xi \in W(\mathbb{R})$, such that the following hold:*

- (1) $W(\mathbb{R})$ is Zariski dense in W ,
- (2) $\psi(\xi) \in S'$,
- (3) ψ is smooth at ξ ,
- (4) for every real closed field $R \supseteq \mathbb{R}$ and every $f \in L_R + R1 \subseteq R[V]$ with $f \geq 0$ on S_R , the pull-back $\psi_R^*(f) \in R[W]$ is a sum of squares in $R[W]$.

(In (4) we have written $L_R = L \otimes R$, which is a finite-dimensional R -linear subspace of $\mathbb{R}[V] \otimes R = R[V]$.)

Proof. By Corollary 3.16, there exists a morphism $\phi: X \rightarrow V$ of affine \mathbb{R} -varieties with $S \subseteq \phi(X(\mathbb{R}))$ such that, for every real closed $R \supseteq \mathbb{R}$ and every $f \in L_R + R1 \subseteq R[V]$ with $f \geq 0$ on S_R , the pull-back $\phi_R^*(f)$ is a sum of squares in $R[X]$. Using the argument of Remark 3.15, we can find a closed irreducible subvariety X' of X satisfying $\phi(X') \subseteq V'$ and $\dim(X') = \dim(V')$, for which $S' \cap \phi(X'(\mathbb{R}))$ is Zariski dense in V' . The restriction $\phi': X' \rightarrow V'$ of ϕ is a dominant morphism between irreducible \mathbb{R} -varieties of the same dimension. By Proposition 4.2, there is a non-empty open affine subset W of X' such that the restriction $\phi'|_W: W \rightarrow V'$ of ϕ' is smooth. Writing $Z = X' \setminus W$ we have $\dim(Z) < \dim(V')$, so the set $\phi'(Z(\mathbb{R}))$ is not Zariski dense in V' . Therefore $S' \cap \phi'(W(\mathbb{R}))$ is still Zariski dense in V' . In particular, we

can find $\xi \in W(\mathbb{R})$ such that $\eta := \phi'(\xi)$ lies in S' . Then it is clear that (1)–(4) are satisfied for $\psi := \phi'|_W: W \rightarrow V'$ and ξ . \square

The first construction is very easy and works for convex hulls of suitable sets of dimension ≥ 3 . First recall:

Lemma 4.7. *Let A be a regular local \mathbb{R} -algebra, let p_1, \dots, p_d be a regular system of parameters of A . If $f(x_1, \dots, x_d)$ is a form in d variables over \mathbb{R} that is not a sum of squares of forms, then $f(p_1, \dots, p_d) \in A$ is not a sum of squares in A .*

The proof uses the associated graded ring of A , see [24], proof of Proposition 6.1.

Lemma 4.8. *Assume that (*) as in 4.5 is given. If for every $\eta \in S'$ there exists $f \in L + \mathbb{R}1 \subseteq \mathbb{R}[V]$ with $f|_S \geq 0$ such that f is not a sum of squares in $\widehat{\mathcal{O}}_{V,\eta}$, then the closed convex hull $\text{conv}(\varphi_L(S))$ is not an sdp set in $\mathbb{A}_L(\mathbb{R}) \cong \mathbb{R}^m$ ($m = \dim(L)$).*

Proof. Assume that the closed convex hull is an sdp set. Then there exists $\psi: W \rightarrow V'$ together with $\xi \in W(\mathbb{R})$ as in Lemma 4.6. Let $\eta = \psi(\xi) \in S'$, and let $f \in L + \mathbb{R}1$ be as in 4.8 for the given η . On the one hand, $\psi^*(f) \in \mathbb{R}[W]$ should be a sum of squares in $\mathbb{R}[W]$, by property (4) of ψ in 4.6. On the other hand, since ψ is smooth in ξ , this contradicts Lemma 4.4, by the choice of f . \square

Example 4.9. Let $x = (x_1, x_2, x_3)$ and put $L = \{f \in \mathbb{R}[x]: \deg(f) \leq 6, f(0) = 0\}$, a linear subspace of $\mathbb{R}[x]$ with $\dim(L) = 83$. For every $\xi \in \mathbb{R}^3$ there exists $f \in L + \mathbb{R}1$ with $f \geq 0$ on \mathbb{R}^3 such that f is not a sum of squares in $\widehat{\mathcal{O}}_{\mathbb{A}^3,\xi}$. For instance, one may take $f = p(x_1 - \xi_1, x_2 - \xi_2, x_3 - \xi_3)$ where p is any ternary sextic form that is psd but not a sum of squares (e.g., the Motzkin form), and use 4.7. Let $\varphi = \varphi_L: \mathbb{A}^3 \rightarrow \mathbb{A}_L \cong \mathbb{A}^{83}$ be the Veronese type embedding associated with L . For any semi-algebraic set $S \subseteq \mathbb{R}^3$ with non-empty interior, it follows from Lemma 4.8 that the closed convex hull of $\varphi(S)$ in \mathbb{R}^{83} is not an sdp set.

Example 4.10. Similarly, let $x = (x_1, x_2, x_3, x_4)$ and $L = \{f \in \mathbb{R}[x]: \deg(f) \leq 4, f(0) = 0\}$. Then $\dim(L) = 69$. Using psd, non-sos quartic forms in four variables and proceeding similarly as in 4.9, we find that the closed convex hull of $\varphi_L(S)$ in \mathbb{R}^{69} is not an sdp set, for any semi-algebraic set $S \subseteq \mathbb{R}^4$ with nonempty interior.

Remark 4.11. The reasoning in the preceding examples was still very coarse. With a finer look we arrive at constructions that are considerably more parsimonious. For example, if in 4.9 we work with the Motzkin form p , we can find a linear system $L \subseteq \mathbb{R}[x]$ of dimension $\dim(L) = 27$ such that $p(x - a) \in \mathbb{R}1 + L$ for every $a \in \mathbb{R}^3$, resulting in an embedding $\mathbb{R}^3 \rightarrow \mathbb{R}^{27}$ with the property of 4.9. Similarly, if in 4.10 we work with the Choi-Lam form $p(x) = x_1^2 x_2^2 + x_2^2 x_3^2 + x_3^2 x_1^2 + x_4^4 - 4x_1 x_2 x_3 x_4$, we can find a linear system of dimension 19 with the desired property. Finding psd, non-sos forms with associated linear systems of small dimension seems to be an interesting problem that is related to apolarity. We plan to get back to this question at a later point.

Remark 4.12. The approach to non-sdp convex sets through Lemma 4.8 did not employ the full strength of the “only if” part of Theorem 3.14. Indeed, it was nowhere used that pull-backs of psd linear polynomials are *uniformly* sums of squares in $\mathbb{R}[X]$ (see Remark 3.17). In turn, the argumentation in 4.9–4.11 applies only to convex hulls of sets of dimension at least three. We now refine the construction, which will provide us with convex hulls of two-dimensional sets without a semidefinite representation.

4.13. Let R be a real closed field containing \mathbb{R} . In the sequel, we always denote the canonical valuation ring of R by B , so $B = \{a \in R : \exists n \in \mathbb{N} |a| < n\}$. The maximal ideal of B is $\mathfrak{m}_B = \{a \in R : \forall n \in \mathbb{N} |na| < 1\}$, the residue field of B is $B/\mathfrak{m}_B = \mathbb{R}$. The reduction map $B \rightarrow B/\mathfrak{m}_B = \mathbb{R}$ will be denoted $a \mapsto \bar{a}$. Nonzero elements in \mathfrak{m}_B will be called *infinitesimals* of R .

Let V be an affine \mathbb{R} -variety. We write $B[V] := \mathbb{R}[V] \otimes B$ (tensor product over \mathbb{R}). If $\phi: X \rightarrow V$ is a morphism of affine \mathbb{R} -varieties, then ϕ_R^* (resp. ϕ_B^*) denotes the induced homomorphism $R[V] \rightarrow R[X]$ (resp. $B[V] \rightarrow B[X]$). Given $\xi \in V(\mathbb{R})$, let $M_{V,\xi} \subseteq B[V]$ be the kernel of the evaluation map $B[V] \rightarrow B$, $f \mapsto f(\xi)$.

We start with several auxiliary results. The following lemma is straightforward:

Lemma 4.14. *Let V be an affine \mathbb{R} -variety, let $\xi \in V(\mathbb{R})$, and let R, B as in 4.13. Then for every $N \geq 1$ the natural map*

$$(\mathcal{O}_{V,\xi}/(\mathfrak{m}_{V,\xi})^N) \otimes B \rightarrow B[V]/(M_{V,\xi})^N$$

of B -algebras is an isomorphism. \square

Lemma 4.15. *Let R, B as in 4.13, and let X be an affine \mathbb{R} -variety for which $X(\mathbb{R})$ is Zariski dense in X . If $g_1, \dots, g_r \in R[X]$ are such that $\sum_{i=1}^r g_i^2$ lies in $B[X]$, then $g_i \in B[X]$ for every i .*

Proof. We can assume $g_i \neq 0$ for every i . Let $f := \sum_{i=1}^r g_i^2$. There is $0 \neq c \in R$ such that $cg_i \in B[X]$ for every i and $\overline{cg_j} \neq 0$ in $(B/\mathfrak{m}_B)[X] = \mathbb{R}[X]$ for at least one index j . It follows that $c^2 f \in B[X]$, and moreover $\overline{c^2 f} = \sum_i (\overline{cg_i})^2$ is nonzero in $(B/\mathfrak{m}_B)[X]$, since $\mathbb{R}[X]$ is real reduced by assumption. Hence $c \notin \mathfrak{m}_B$, which means that $\frac{1}{c} \in B$, and so indeed $g_i \in B[X]$ for every index i . \square

Lemma 4.16. *Let R, B as in 4.13, and let $\phi: X \rightarrow V$ be a morphism of affine \mathbb{R} -varieties. Assume that $X(\mathbb{R})$ is Zariski dense in X , and that ϕ is smooth at $\xi \in X(\mathbb{R})$. If $f \in B[V]$ and $N \geq 1$ are such that f is not a sum of squares in $B[V]$ modulo $(M_{V,f(\xi)})^N$, then $\phi_R^*(f) \in R[X]$ is not a sum of squares in $R[X]$.*

Proof. Write $\eta = \phi(\xi)$. By Proposition 4.3, the smoothness assumption implies that the completed local ring $\widehat{\mathcal{O}}_{X,\xi}$ is $\widehat{\mathcal{O}}_{V,\eta}$ -isomorphic to a power series ring over $\widehat{\mathcal{O}}_{V,\eta}$. In particular, this implies that $\phi^*: \mathcal{O}_{V,\eta}/(\mathfrak{m}_{V,\eta})^N \rightarrow \mathcal{O}_{X,\xi}/(\mathfrak{m}_{X,\xi})^N$ has a retraction, i.e. there is a homomorphism $\rho: \mathcal{O}_{X,\xi}/(\mathfrak{m}_{X,\xi})^N \rightarrow \mathcal{O}_{V,\eta}/(\mathfrak{m}_{V,\eta})^N$ for which the composition $\rho \circ \phi^*$ is the identity on $\mathcal{O}_{V,\eta}/(\mathfrak{m}_{V,\eta})^N$. Tensoring with B and using Lemma 4.14 gives the commutative diagram

$$\begin{array}{ccc} B[V] & \xrightarrow{\phi_B^*} & B[X] \\ \downarrow & & \downarrow \\ B[V]/(M_{V,\eta})^N & \longrightarrow & B[X]/(M_{X,\xi})^N \end{array}$$

whose bottom map has a retraction. From the hypothesis it therefore follows that $\phi_B^*(f) \in B[X]$ cannot be a sum of squares in $B[X]$. By Lemma 4.15, $\phi_R^*(f)$ is not a sum of squares in $R[X]$ either. \square

Lemma 4.17. *Let R, B as in 4.13, let V be an affine \mathbb{R} -variety, and let $\xi \in V(\mathbb{R})$ be a nonsingular \mathbb{R} -point. If $u_1, \dots, u_d \in \mathbb{R}[V]$ form a regular parameter sequence of V at ξ , we have an isomorphism*

$$B[V]/(M_{V,\xi})^N \cong B[x_1, \dots, x_d]/\langle x_1, \dots, x_d \rangle^N$$

of B -algebras which makes the cosets of u_i and x_i correspond to each other for $i = 1, \dots, d$.

Proof. Clear from the isomorphism $\mathbb{R}[[x_1, \dots, x_d]] \rightarrow \widehat{\mathcal{O}}_{V, \xi}$ sending x_i to u_i , and from Lemma 4.14. \square

The next result implies that, for $R \neq \mathbb{R}$ and $n \geq 2$, there exist psd polynomials in $B[x]$ that are not sums of squares modulo $\langle x \rangle^N$ for large N , where $x = (x_1, \dots, x_n)$.

Proposition 4.18. *Let $f \in \mathbb{R}[x_0, x] = \mathbb{R}[x_0, \dots, x_n]$ be homogeneous of degree d , and assume that f is not a sum of squares in $\mathbb{R}[x_0, x]$. Let R, B be as in 4.13. If $\epsilon \neq 0$ is an infinitesimal in R , the polynomial $f(\epsilon, x) \in B[x]$ is not a sum of squares in $B[x]$ modulo $\langle x_1, \dots, x_n \rangle^{d+1} B[x]$.*

Proof. Assume we have an identity $f(\epsilon, x) + g(x) = \sum_j p_j(x)^2$ where $g(x) \in \langle x \rangle^{d+1} B[x]$ and $p_j(x) \in B[x]$. Replacing x by ϵx yields

$$\epsilon^d f(1, x) + g(\epsilon x) = \sum_j p_j(\epsilon x)^2. \quad (*)$$

The left hand side is divisible by ϵ^d in $B[x]$. By Lemma 4.15, the polynomial $q_j(x) := \epsilon^{-d/2} p_j(\epsilon x) \in R[x]$ lies in $B[x]$ for every j . Putting $g'(x) = \epsilon^{-(d+1)} g(\epsilon x)$ we have $g'(x) \in B[x]$, therefore dividing $(*)$ by ϵ^d gives

$$f(1, x) + \epsilon g'(x) = \sum_j q_j(x)^2,$$

an identity in $B[x]$. Reducing coefficient-wise modulo \mathfrak{m}_B implies that $f(1, x)$ is a sum of squares in $\mathbb{R}[x]$, contradicting the hypothesis. \square

Lemma 4.19. *Assume that $(*)$ as in 4.5 is given, and assume that $R \supseteq \mathbb{R}$, $R \neq \mathbb{R}$ is a real closed field with canonical valuation ring B (4.13). For every $\eta \in S'$ assume that there exists $f \in L_B + B1 \subseteq B[V]$ with $f \geq 0$ on S_R such that f is not a sum of squares in $B[V]/(M_{V, \eta})^N$ for some $N \geq 1$. Then the closed convex hull $\text{conv}(\varphi_L(S))$ is not an sdp set in $\mathbb{A}_L(\mathbb{R}) \cong \mathbb{R}^m$ ($m = \dim(L)$).*

(Here $L_B := L \otimes B \subseteq \mathbb{R}[V] \otimes B = B[V]$.)

Proof. Assume that the closed convex hull is an sdp set. Then there exists $\psi: W \rightarrow V'$ together with $\xi \in W(\mathbb{R})$, as in Lemma 4.6. Let $\eta = \psi(\xi) \in S'$, and let $f \in L_B + B1$ as in 4.19 for the given η . On the one hand, $\psi_R^*(f) \in R[W]$ should be a sum of squares in $R[W]$, by property (4) of ψ in 4.6. On the other hand, $\psi_R^*(f)$ is not a sum of squares in $R[W]$ by Lemma 4.16. This contradiction proves Lemma 4.19. \square

Lemma 4.20. *Let R, B be as in 4.13, and assume $R \neq \mathbb{R}$. Let V be an affine \mathbb{R} -variety, let $\eta \in V_{\text{reg}}(R)$, and let $q_1, \dots, q_n \in B[V]$ be a regular parameter sequence for $\mathcal{O}_{V, \eta}$. Moreover let $f \in \mathbb{R}[x_0, \dots, x_n]$ be a form that is psd but not a sum of squares. If $\epsilon \neq 0$ is any infinitesimal in R , then $f(\epsilon, q_1, \dots, q_n) \in B[V]$ is psd on $V(R)$, but is not a sum of squares in $B[V]/(M_{V, \eta})^N$ for $N > \deg(f) + 1$.*

Proof. Put $p := f(\epsilon, q_1, \dots, q_n)$. It is clear that $p \geq 0$ on $V(R)$. Let $x = (x_1, \dots, x_n)$. We have an isomorphism $B[x]/\langle x \rangle^N \xrightarrow{\sim} B[V]/(M_{V, \eta})^N$ for every $N \geq 1$ that sends x_i to q_i ($i = 1, \dots, n$) (Lemma 4.17). It maps the residue class of $f(\epsilon, x)$ to the residue class of p . By Proposition 4.18, this element (in either ring) is not a sum of squares when $N > \deg(f)$. \square

Example 4.21. Let $x = (x_1, x_2)$, and put $L = \{f \in \mathbb{R}[x] : \deg(f) \leq 6, f(0) = 0\}$, a linear subspace of $\mathbb{R}[x]$ of dimension 27. Consider the associated embedding $\phi_L : \mathbb{A}^2 \rightarrow \mathbb{A}_L \cong \mathbb{A}^{27}$. If $S \subseteq \mathbb{R}^2$ is any semi-algebraic set with non-empty interior, the closed convex hull of $\phi_L(S)$ in \mathbb{R}^{27} is not an sdp set. Indeed, choose a sextic form $p \in \mathbb{R}[x_0, x_1, x_2]$ that is psd but not a sum of squares, and let $0 \neq \epsilon$ be an infinitesimal of R . Given $\xi \in S$, the polynomial $f := p(\epsilon, x_1 - \xi_1, x_2 - \xi_2)$ in $B[x]$ lies in $L_B + B1$, and f is not a sum of squares in $B[x]/(M_{\mathbb{A}^2, \xi})^7$ by Proposition 4.18 and Lemma 4.17. It follows from Lemma 4.20 that $\overline{\text{conv}(\phi_L(S))}$ is not an sdp set.

Remark 4.22. Similar to Remark 4.11, we can arrive at examples of smaller dimension when we take a finer look. For instance, consider the sextic form $p = x_0^6 + x_0^5 x_1 + (x_0 x_2^2 - x_1^3)^2$ in $\mathbb{R}[x_0, x_1, x_2]$. It is easy to see that p is psd but not a sum of squares. For this choice of p , it suffices to take a suitable linear subspace $L \subseteq \mathbb{R}[x, y]$ with $\dim(L) = 16$. Therefore, the closed convex hull of $\phi_L(S)$ in \mathbb{R}^{16} is not an sdp set, for any semi-algebraic $S \subseteq \mathbb{R}^2$ with non-empty interior.

Theorem 4.23. *Let $S \subseteq \mathbb{R}^m$ be any semi-algebraic set with $\dim(S) \geq 2$. Then, for some k , there exists a polynomial map $\varphi : S \rightarrow \mathbb{R}^k$ such that the closed convex hull of $\varphi(S)$ in \mathbb{R}^k is not semidefinitely representable.*

Proof. Let $V \subseteq \mathbb{A}^m$ be the Zariski closure of S . Fix a point $\xi \in S \cap V_{\text{reg}}(\mathbb{R})$ such that $\dim_{\xi}(S) \geq 2$ and S contains an open neighborhood of ξ in $V(\mathbb{R})$. Let $p_1, \dots, p_n \in \mathbb{R}[V]$ ($n \geq 2$) be a regular sequence of parameters for $\mathcal{O}_{V, \xi}$. Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ be tuples of variables, let $f \in \mathbb{R}[t, x]$ be a form in $n+1$ variables that is psd but not a sum of squares, and put $d = \deg(f)$. We can write

$$f(t, x + y) = \sum_{i=0}^d g_i(x) h_{d-i}(t, y)$$

where $g_i \in \mathbb{R}[x]$ and $h_i \in \mathbb{R}[t, y]$ are forms of degree i ($i = 0, \dots, d$). There is a Zariski open neighborhood $U \subseteq V_{\text{reg}}$ of ξ such that, for any $\eta \in U(\mathbb{R})$, the sequence $p_i - p_i(\eta)$ ($i = 1, \dots, n$) is a regular sequence of parameters for $\mathcal{O}_{V, \eta}$. Let $L \subseteq \mathbb{R}[V]$ be a finite-dimensional linear subspace that contains $g_i(p_1, \dots, p_n)$ for $i = 1, \dots, d$, and choose a real closed field R properly containing \mathbb{R} . For any $a = (a_0, \dots, a_n) \in B^{n+1}$, the element $q_a := f(a_0, p_1 + a_1, \dots, p_n + a_n) \in B[V]$ lies in $L_B + B1$ and satisfies $q_a \geq 0$ on $V(R)$. Let $\eta \in U(\mathbb{R})$, and put $a = (\epsilon, -p_1(\eta), \dots, -p_n(\eta)) \in B^{n+1}$ where $\epsilon \neq 0$ is infinitesimal in R . Then, by Lemma 4.20, $q_a \in B[V]$ is nonnegative on $V(R)$, and q_a is not a sum of squares in $B[V]/(M_{V, \eta})^{d+1}$. By Lemma 4.19, this shows that $\overline{\text{conv}(\varphi(S))}$ is not an sdp set. \square

The previous examples already indicate that convex hulls of Veronese sets are typically not semidefinitely representable. Specifically, we have:

Corollary 4.24. *Let n, d be positive integers with $n \geq 3$ and $d \geq 4$, or with $n = 2$ and $d \geq 6$. Let m_1, \dots, m_N be the non-constant monomials of degree $\leq d$ in (x_1, \dots, x_n) (so $N = \binom{n+d}{n} - 1$). Then for any semi-algebraic set $S \subseteq \mathbb{R}^n$ with non-empty interior, the closed convex hull of*

$$v(S) := \left\{ (m_1(\xi), \dots, m_N(\xi)) : \xi \in S \right\}$$

in \mathbb{R}^N has no semidefinite representation.

Proof. It suffices to apply Lemma 4.19 and Proposition 4.18. \square

For positive integers n, d let $\Sigma_{n,2d}$ (resp. $P_{n,2d}$) denote the cone of all degree $2d$ forms in $\mathbb{R}[x_1, \dots, x_n]$ that are sums of squares of forms (resp. that are positive semidefinite).

Corollary 4.25. *The psd cone $P_{n,2d}$ is semidefinitely representable only in the cases where $P_{n,2d} = \Sigma_{n,2d}$, i.e. only for $2d = 2$ or $n \leq 2$ or $(n, 2d) = (3, 4)$.*

Proof. It is well-known and easy to see that the dual $\Sigma_{n,2d}^*$ of the sos cone is a spectrahedral cone. Therefore $\Sigma_{n,2d}$, being closed, is an sdp cone. Let n, d be such that $\Sigma_{n,2d} \neq P_{n,2d}$. The dual cone $P_{n,2d}^*$ can be identified with the convex (or conical) hull of the image of the degree $2d$ Veronese map

$$v_{n,2d}: \mathbb{R}^n \rightarrow \mathbb{R}^N, \quad \xi \mapsto (\xi^\alpha)_{|\alpha|=2d}$$

where $N = \binom{n+2d-1}{n-1}$ is the number of monomial of degree $2d$ in (x_1, \dots, x_n) . By 4.24, a suitable affine hyperplane section of this cone fails to be an sdp set. So $P_{n,2d}^*$ itself cannot be an sdp cone, and therefore neither can be $P_{n,2d}$. \square

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